2. FORMULATION: THE CHARACTERISTIC FUNCTION

In this section we obtain the general forms of the first-order characteristic functions (c.f.'s) and probability densities (p.d.'s) and distributions (P.D.'s) for the "impulsive" interference of the various man-made and natural sources described in Part I above.

Our first step is to derive the desired general forms of the characteristic functions for the envelope of the received wave. The next step is to take advantage of the various physical conditions of the model, further to reduce our results to the particular expressions appropriate to the Class A and Class B interference, which can then be put in forms suitable for evaluation. A number of important parameters of these interference processes appear in the analysis and have important physical implications, which we shall develop further in the subsequent sections.

2.1 The Basic Statistical Model:

We assume as before [Middleton, 1974] for our basic model that there is an infinite number of potential sources in a source domain Λ, and that while the basic waveforms emitted all have the same form, their scale, durations, frequencies, etc., may be randomly distributed. Our fundamental postulate of this basic interference model is that: (i), the locations of the various possible emitting sources are poisson distributed in space; (ii), the emission times of the possible sources are similarly poisson distributed in time. Physically, this means that the sources are statistically independent, both in location and emission. Thus, by a slight generalization of earlier results [Middleton; 1967, 1972b, 1974], we can write for the first-order characteristic function of the instantaneous amplitude, X, of the received interference process

$$F_1(i\xi)_X = \exp\left\{\left\langle \int_{\Lambda,\hat{\epsilon}} \rho(\underset{\sim}{\lambda},\hat{\epsilon}) \left\langle e^{i\xi U(t;\underset{\sim}{\lambda},\underset{\sim}{\hat{\xi}},\dots)}-1 \right\rangle_{\underset{\sim}{\theta}} d\underset{\sim}{\lambda}\, d\hat{\epsilon}\right\rangle\right\}. \qquad (2.1)$$

Here $\hat{\epsilon}$ is an epoch, indicating vis-à-vis the receiver's (i.e. observer's) time t when a source may emit. The $\underset{\sim}{\lambda} = (\lambda,\theta,\phi)$ are coördinates, or a vector magnitude, appropriate to the geomtry of the source field, located in the region Λ, and of the receiver, with $d\underset{\sim}{\lambda}$ (= $d\lambda d\phi$) for a surface element; (= $d\lambda d\theta d\phi$) for a volume element. The quantity $\rho(\underset{\sim}{\lambda},\hat{\epsilon})$ is the "process density" of this joint space-time poisson interference process, and is non-negative, and can be regarded as proportional to a probability density [cf. (2.28) below]. The $\langle\ \rangle_{\underset{\sim}{\theta}}$ denotes a statistical average, e.g. $\int_{[\underset{\sim}{\theta}]}[\ ]w_1(\underset{\sim}{\theta})d\underset{\sim}{\theta}$, over various random parameters ($\underset{\sim}{\theta}$) which may be pertinent to our source model, such as doppler, source amplitude and duration, etc.

The U are the typical waveforms of the emitting sources, after reception by the (assumed linear) aperture - RF - IF stages of our "narrow-band" receiver. The received process X is given by

$$X(t) = \int_{\underset{\sim}{Z}(=\underset{\sim}{\Lambda}\times\underset{\sim}{\theta})} U(t|\underset{\sim}{Z})dN(\underset{\sim}{Z}), \tag{2.2}$$

where the {dN} are a poisson point process (in $\underset{\sim}{\lambda}$ and $\hat{\varepsilon}$), such that

$$\langle X(t)\rangle = \int_{\underset{\sim}{Z}} U\langle dN\rangle = \left\langle \int_{\Lambda} \rho(\underset{\sim}{\lambda},\hat{\varepsilon})U(t;\underset{\sim}{\lambda},\hat{\varepsilon},\underset{\sim}{\theta})d\underset{\sim}{\lambda}\ d\hat{\varepsilon}\right\rangle_{\underset{\sim}{\theta}}, \tag{2.3a}$$

is the process mean (if any), and

$$\langle X(t_1)X(t_2)\rangle = \iint_{\underset{\sim}{Z}^2} U_1U_2\langle dN_1dN_2\rangle = \left\langle \int_{\Lambda} \rho(\underset{\sim}{\lambda},\hat{\varepsilon})U(t_1;\underset{\sim}{\lambda},\hat{\varepsilon},\underset{\sim}{\theta})U(t_2;\underset{\sim}{\lambda},\hat{\varepsilon},\underset{\sim}{\theta})d\underset{\sim}{\lambda}d\hat{\varepsilon}\right\rangle_{\underset{\sim}{\theta}}$$

$$+ \langle X(t_1)\rangle\langle X(t_2)\rangle \tag{2.3b}$$

is the general second-moment of X(t), under our basic poisson assumption above of source location and emission.* Higher moments may be similarly obtained.

Since we are interested here in the envelope of the received process X, which is always narrow-band, in as much as the receiver is itself narrow-band, we have to consider the new random variables X_c, X_s, representing the slowly-varying "in-phase" and "out-of-phase" components of X, viz.,

$$X(t) = X_c(t)\cos\omega_0 t + X_s(t)\sin\omega_0 t = \mathrm{Re}\left\{(X_c - iX_s)e^{i\omega_0 t}\right\} = \mathrm{Re}\left\{\hat{X}_0 e^{i\omega_0 t}\right\} \tag{2.4}$$

$$= \mathrm{Re}\left\{E\ e^{i(\omega_0 t-\psi)}\right\}, \tag{2.4a}$$

* For a general development of process statistics, not necessarily limited to the poisson case of independent sources, see, for example, recent work [Middleton,1974,1975b] in the development of generalized scattering models.

where now $\omega_o(=2\pi f_o)$ is the central (angular) frequency of the final ($\equiv$IF) stage of the receiver, and

$$E = \sqrt{X_c^2+X_s^2} \quad ; \quad \psi = \tan^{-1}(X_s/X_c) \quad ; \quad \therefore \hat{X}_o = X_c - iX_s = Ee^{-i\omega_o\psi} \tag{2.4b}$$

with

$$X_c = E\cos\psi, \quad X_s = E\sin\psi .$$

Here E, ψ are, respectively the envelope and phase of the narrow band received process X. In functional form, cf. (2.2), we can write alternatively

$$X(t) = \mathrm{Re}\left\{\int_{\underset{\sim}{Z}} [U_c(t|\underset{\sim}{Z}) - iU_s(t|\underset{\sim}{Z})]e^{i\omega_o t} dN(\underset{\sim}{Z})\right\} \text{, or} \tag{2.5a}$$

$$= \mathrm{Re}\left\{\int_{\underset{\sim}{Z}} e(t|\underset{\sim}{Z})e^{i\omega_o t - i\phi_s(t|\underset{\sim}{Z})} dN(\underset{\sim}{Z})\right\} , \tag{2.5b}$$

in terms of an envelope and phase, where

$$e(t|\underset{\sim}{Z}) \equiv \sqrt{U_c^2+U_s^2} \quad ; \quad \phi_s \equiv \tan^{-1}(U_s/U_c). \tag{2.5c}$$

Comparing (2.5) and (2.4) we see at once that

$$(X_c, X_s) = \int_{\underset{\sim}{Z}} (U_c, U_s) dN(\underset{\sim}{Z}) \quad ; \tag{2.6a}$$

$$E\, e^{-i\psi_s} = \int_{\underset{\sim}{Z}} e(t|\underset{\sim}{Z})e^{-i\phi_s(t|\underset{\sim}{Z})} dN(\underset{\sim}{Z}) . \tag{2.6b}$$

The characteristic function which we need now is for the random variables X_c, X_s, namely,

$$F_1(i\xi, i\eta)_{X_c,X_s} = \exp\left\{\left\langle \int_\Lambda \rho(\underset{\sim}{\lambda}, \hat{\varepsilon})[e^{i\xi U_c + i\eta U_s} - 1]\right\rangle \underset{\sim}{d\lambda} d\hat{\varepsilon}\right\} , \tag{2.7}$$

which is the two-dimensional generalization of (2.1) required here. The corresponding p.d. is

$$W_1(X_c, X_s) = \iint_{-\infty}^{\infty} F_1(i\xi, i\eta)_{X_c,X_s}\, e^{-i\xi X_c - i\eta X_s} d\xi d\eta/(2\pi)^2 . \tag{2.8}$$

Also, we have, formally, the following expression for the joint first-order density of envelope E and phase ψ, in terms of the in-phase and out-of-phase components X_c, X_s of X.

$$w_1(E,\psi) = W_1(X_c,X_s)\left|\frac{\partial(X_c,X_s)}{\partial(E,\psi)}\right| = EW_1(E\cos\psi, E\sin\psi), \quad \left.\begin{matrix} E \geq 0 \\ 0 \leq \psi < 2\pi \end{matrix}\right\}, \tag{2.9}$$

where W_1, (2.8), is now

$$W_1(E\cos\psi, E\sin\psi) = \iint_{-\infty}^{\infty} F_1(i\xi,i\eta)_{X_c,X_s} e^{-iE(\xi\cos\psi+\eta\sin\psi)} d\xi d\eta/(2\pi)^2, \tag{2.9a}$$

with F_1 therein given by (2.7)

To proceed further, we make use of a number of results from our earlier development of the physical model [Sec. 2.2, Middleton, 1974], to write for the narrow-band basic waveform U (at the output of the receiver's IF)

$$U = U_{nb} = B_0(t,\underset{\sim}{\lambda}|\hat{\varepsilon},\underset{\sim}{\theta})\cos\mu_d\Psi(t,\underset{\sim}{\lambda}|\hat{\varepsilon},\underset{\sim}{\theta}), \quad \mu_d = 1+\varepsilon_d, \tag{2.10}$$

where B_0 (>0) is an envelope, whose detailed structure we shall consider in more detail later and where Ψ is a phase, which has the form

$$\Psi \equiv \omega_0(t-\lambda-\hat{\varepsilon}) - \mu_d^{-1}[\Phi_S(t-\lambda-\hat{\varepsilon},\underset{\sim}{\theta})+\phi_T(\underset{\sim}{\lambda},f_0)+\phi_R(\underset{\sim}{\lambda},f_0)] , \tag{2.10a}$$

in which Φ_S, ϕ_T, ϕ_R are respectively the typical source phase, and the phase angles of the source (T) and receiver (R), complex beam patterns [cf. Sec. 2.5 below]. [The quantity $\varepsilon_d(=\mu_d-1)$ is the sum of the relative dopplers between sources and the receiver, and is always small, $0(10^{-5,-6})$, in our applications, viz: $\varepsilon_d = 2v/c = 0(10^{-6})$ for $v = 0(10^5$ mph) so that the envelope B_0 is independent of ε_d.

From the fact that $U_{nb} = U_c\cos\omega_0 t+U_s\sin\omega_0 t$, we see at once from (2.10) that

$$U_c = B_0\cos[\Phi_S'+\mu_d\omega_0(\lambda+\hat{\varepsilon})-\varepsilon_d\omega_0 t]; \; U_s = B_0\sin[\Phi_S'+\mu_d\omega_0(\lambda+\hat{\varepsilon})-\varepsilon_d\omega_0 t], \tag{2.11}$$

where $\Phi_s' \equiv \Phi_s+\phi_T+\phi_R$. We now use polar coördinates

$$\xi = r\cos\phi \quad ; \quad \eta = r\sin\phi \; ; \quad |\partial(\xi,\eta)/\partial(r,\phi)| = r \tag{2.12a}$$

and transform from (ξ,η)-space to (r,ϕ)-space in (2.9a). Thus, we see that

$$d\xi d\eta \, F_1(i\xi,i\eta)_{X_c;X_s} = r\,\hat{F}_1(ir,\phi)drd\phi \;, \quad 0 < r < \infty; \quad 0 \leq \phi < 2\pi. \tag{2.12b}$$

The c.f. $\hat{F}_1$ is $F_1(ir\cos\phi, ir\sin\phi)_{X_c,X_s}$, (2.7), which with (2.11) now reduces explicitly to

$$\boxed{\hat{F}_1(ir,\phi) = \exp\left\{\left\langle\int_\Lambda \rho(\underset{\sim}{\lambda},\hat{\varepsilon})e^{irB_0\cos[\Phi_s'+\mu_d\omega_0(\lambda+\hat{\varepsilon})-\varepsilon_d\omega_0 t-\phi]}-1\right\rangle_{\underset{\sim}{\theta}} d\underset{\sim}{\lambda}d\hat{\varepsilon}\right\}.} \tag{2.13}$$

The first-order p.d. for the envelope and phase (E,ψ), (2.9), with the help of (2.12b), (2.9a), becomes

$$\boxed{w_1(E,\psi) = E\int_0^\infty rdr\int_0^{2\pi} \frac{d\phi}{(2\pi)^2}\,\hat{F}_1(ir,\phi)e^{-iEr\cos(\psi-\phi)}}$$

$$E > 0, \; 0 \leq \psi < 2\pi \,. \tag{2.14}$$

This is as far as we can go without further appeal to the physical model, in particular, to the statistics governing the locations $(\sim\underset{\sim}{\lambda})$ of the sources and the epoch $(\hat{\varepsilon})$ of interference emissions. We note, however, that the p.d. for the envelope alone is readily found, e.g.* the integration over ψ (in $0,2\pi$), is well-known [cf. (2.19) following)]:

$$\boxed{W_1(E) = \int_0^{2\pi} w_1(E,\psi)d\psi = E\int_0^\infty rJ_0(rE)dr\int_0^{2\pi}\hat{F}_1(ir,\phi)d\phi/2\pi.} \tag{2.15}$$

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* As usual, functions of different arguments are different functions, e.g. $W_1(E)\neq W_1(\psi)\neq W_1(X_c,X_s)$, etc., unless it is otherwise stated.

In addition, we have respectively for the P.D., and exceedance probability, or APD (a posterior probability here, that E exceeds a level $E_0(>0)$) defined as usual by

$$D_1(E_0) \equiv \int_0^{E_0} W_1(E)dE \quad ; \; P_1(E \geq E_0) \equiv \int_{E_0}^{\infty} W_1(E)dE = 1-D(E_0), \tag{2.16}$$

the following results, where we have used

$$\int_0^z zJ_0(z)dz = zJ_1(z), \tag{2.16a}$$

viz:

$$\boxed{\begin{aligned} D_1(E_0) &= E_0\int_0^{\infty} J_1(rE_0)dr\int_0^{2\pi} \hat{F}_1(ir,\phi)d\phi/2\pi, \qquad E_0 \geq 0 & (2.17a) \\ P_1(E \geq E_0) &= 1-E_0\int_0^{\infty} J_1(rE_0)dr\int_0^{2\pi} \hat{F}_1(ir,\phi)d\phi/2\pi \; . & (2.17b) \end{aligned}}$$

Our results (2.13)-(2.17) are generalizations of earlier results [Furutsu and Ishida, 1960; Middleton, 1972b; Giordano, 1970], where our basic assumptions, so far, postulate only poisson distributions of source location and emissions, e.g. essentially independent sources. No restrictions on the specific character of the statistics of the source parameters are as yet introduced. It is for this reason that the characteristic function $\hat{F}_1$ depends on ϕ, as well as on r.

2.2 First Reduction of the c.f. $\hat{F}_1$: The Narrow-Band Receiver Condition

At this point we invoke certain properties of the basic waveform $B_0\cos[\Phi_s' + \mu_d\omega_0(\lambda+\hat{\varepsilon}) - \omega_0\varepsilon_d t - \phi]$ which appears in the exponent in the integrand of (2.13). We use the facts that (i), B_0, Φ_s, are both slowly-varying functions of λ; and (ii), the process density $\rho(\underset{\sim}{\lambda},\hat{\varepsilon})$ is likewise slowly varying, vis-à-vis cos $\omega_0\mu_d\lambda$, sin $\omega_0\mu_d\lambda$. Employing the familiar expansion in Bessel functions,